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Multi-matrix models and emergent geometry

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Abstract: Encouraged by the AdS/CFT correspondence, we study emergent local geometry in large N multi-matrix models from the perspective of a strong coupling expansion. By considering various solvable interacting models we show how the emergence or non-emergence of local geometry at strong coupling is captured by observables that effectively measure the mass of off-diagonal excitations about a semiclassical eigenvalue background. We find emergent geometry at strong coupling in models where a mass term regulates an infrared divergence. We also show that our notion of emergent geometry can be usefully applied to fuzzy spheres. Although most of our results are analytic, we have found numerical input valuable in guiding and checking our results.

Keywords: Matrix Models, AdS-CFT Correspondence, 1/N Expansion.

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1. Introduction

The emergence of geometry has long been recognised as a key issue in quantum gravity. The more recent discovery of the AdS/CFT correspondence [1] has indicated a complementary viewpoint: Certain large N field theories are most simply described starting from a higher dimensional dynamical geometry. The geometry is not apparent in the weak coupling (Lagrangian) description of the quantum field theory, and in this sense is emergent. The emergence of spacetime can be thought of as a precise realisation of 't Hooft's insight [2] that large N gauge theories are string theories, together with the fact that string theory describes a theory of quantum geometry.

The best understood cases of emergent geometry from field theory begin with D branes in a pre-existing geometry.[1] The geometry 'dual' to the strongly coupled field theory on the D brane worldvolume is then obtained by computing the gravitational backreaction of the D branes [1, 7]. These theories are often supersymmetric and, in the best understood cases,

[1]Early pre-stringy observations of emergent geometry include for instance [3 – 6].

conformal. One would want a much more comprehensive understanding of how geometry emerges, and the relevant situations where it can be applied. In essence, one needs to find a way to solve a key aspect of the field theory dynamics and find geometry.

There is ample reason to believe that the large N expansion might be relevant for QCD, the theory of the strong interactions, and one would like to have a first principles approach to calculating the corresponding dual geometry (or dual string theory). If this is understood, exploiting the geometrical information may allow us to describe the strong coupling dynamics of the theory in a more economical way. The strong interactions are not supersymmetric and, in the regime of strong coupling, they are not conformal either.

In this paper we would like to study emergent geometry from first principles in a simplified setting. We will study large N systems in zero dimensions, with and without supersymmetry. We will solve the multi-matrix models exactly in certain limits, and look for emergent geometry.

The simplest emergence of geometry from large N matrices occurs in a Gaussian matrix model for a single matrix. In the large N limit, the integral is dominated by a saddle point in which the eigenvalues of the matrix are distributed in a semicircle [8]. This semicircle can be thought of as a continuum geometry that emerges at large N. This very simple example already illustrates an important theme for us. The emergent geometry is possible because the Gaussian mass term balances the repulsive inter-eigenvalue force. The Gaussian model may be generalised by introducing interaction terms for the matrix, and an elegant mathematical theory allows us to find the eigenvalue distribution in that case [9].

A tractable step beyond single matrix models are normal matrix models, in which a matrix and its Hermitian conjugate commute with one another. These models show an emergent geometry at large N which describes a two dimensional droplet [10]. Again there is an elegant mathematical framework to describe these models, see for instance [11, 12].

For general multi-matrix models, even with just two matrices, the question of whether there is or not an emergent geometry is difficult and a developed framework is lacking. Unlike the case of single matrix or normal matrix models, these systems can rarely [13] be solved exactly even at large N. Some recent numerical work can be found in [14 – 19]. However, many of the cases of most interest are numerically problematic because of the sign problem in certain supersymmetric theories.

The AdS/CFT correspondence [1] suggests that a natural organising principle is a strong coupling limit. For the correspondence to work, the degrees of freedom in the field theory need to 'geometrise' at strong coupling in order to reproduce the dual higher dimensional (dynamical) spacetime. There are various discussions of emergent geometry in AdS/CFT in the literature, including [20 – 22]. However, only recently has an attempt been made to systematise the emergence of geometry as a consequence of the strong coupling limit [23].

With multiple matrices, it seems that a key aspect of an emergent classical geometry is that the matrices commute with each other in the large N limit. In this way the typical N^2 degrees of freedom of matrices get effectively reduced to order N degrees of freedom at low energies. The collective description of these low energy degrees of freedom can often be given in terms of a joint eigenvalue distribution for several matrices. It is the geometrical description of this eigenvalue distribution that produces the emergent geometry. One objective of this paper is to make the notion of commuting matrices more precise. In [23] it was proposed that the zero modes of the six scalar fields of $\mathcal{N} = 4$ super Yang-Mills theory on a spatial S^3 commuted at strong coupling. From this proposal one can show that the joint eigenvalue distribution of these six matrices forms an S^5 that should be identified with the geometric S^5 that arises in the $AdS_5 \times S^5$ of the dual IIB string theory.

The proposal of [23] was subsequently generalised to describe orbifolds of the $\mathcal{N} = 4$ theory [24] and also to $\mathcal{N} = 1$ theories [25, 26]. Various successful checks of the proposal were performed in [27, 26, 28]. However, given that the theory involves infinitely many coupled matrices with comparable masses, it is difficult to prove the validity of the truncation to just six fields. Furthermore, the theories considered have all been superconformal and somewhat similar to the maximally symmetric Yang-Mills theory in four dimensions.

In this paper we will look at models in which we can show directly and unambiguously whether the matrices commute or not at strong coupling. The paper is organised as follows. We will solve a succession of multi-matrix models in the strong coupling limit. At each step, we shall check our analytic results with numerics. Although many of our final results are analytic, the interplay with numerics has been a very important guide towards these results. In a later section we consider some models that we cannot solve analytically. Finally, we also show how our notion of emergent geometry can be applied to fuzzy spheres. The concluding discussion includes a summary of our results. The main point we emphasise is that higher dimensional emergent geometry can arise naturally in models where a mass term regularises an infrared divergence of a massles model. In these cases, at strong coupling, the eigenvalues are sufficiently spread out that off-diagonal modes do not contribute to low energy dynamics.

2. A two matrix model at strong coupling

2.1 Solving the model: parabolic distribution

We begin by considering a two matrix model that can and has [29] been solved exactly at all couplings. We could read off most of the quantities we need from [29]. Instead we will solve the model in a more low-tech way, that is geared towards the more general issues we would like to understand at strong 't Hooft coupling. We will see that this model can be re-interpreted as an emergent two dimensional geometry at strong coupling.

Consider the two Hermitian matrix model

$$Z = \int \mathcal{D}X\mathcal{D}Y e^{-\mathrm{tr}X^2 - \mathrm{tr}Y^2 + g^2 \mathrm{tr}[X,Y]^2} . \tag{2.1}$$

Because this integral is quadratic in both X and Y, we can diagonalise X and then perform the Y integral exactly. The partition function becomes

$$Z = \int dx_1 \dots dx_N e^{-\sum_i x_i^2} \prod_{i \neq j} \frac{x_i - x_j}{\sqrt{1 + g^2 (x_i - x_j)^2}} , \tag{2.2}$$

$$= \int dx_1 \dots dx_N e^{-\sum_i x_i^2 + \frac{1}{2}\sum_{i\neq j} \log(x_i - x_j)^2 - \frac{1}{2}\sum_{i \neq j} \log[1 + g^2(x_i - x_j)^2]} . \tag{2.3}$$

The first logarithm appearing here is the standard Vandermonde determinant arising upon diagonalising X. We can look for a large N saddle point to this integral. The saddle point equations of motion are

$$x_i = \sum_{j \neq i} \frac{1}{(x_i - x_j)[1 + g^2(x_i - x_j)^2]} \,. \tag{2.4}$$

These equations have been solved at large N by Kazakov-Kostov-Nekrasov [29], somewhat implicitly. Here we shall be interested in the strong 't Hooft coupling limit $\lambda = g^2 N \gg 1$, and shall take a less sophisticated approach.

In the large N limit, the saddle point equation becomes

$$x = \int \frac{\rho(y) dy}{(x - y)[1 + g^2(x - y)^2]} \,. \tag{2.5}$$

We took the continuum limit $\sum \to \int \rho(x) dx$ and introduced the eigenvalue density $\rho(x)$. We are working with the normalisation $\int \rho(x) dx = N$. As usual, a principal value is understood in the integral (2.5).

We can solve this equation at strong 't Hooft coupling by first noting the general result

$$\mathcal{P} \int_{-1}^{1} \frac{\rho(y) dy}{(x - y)[1 + J^2(x - y)^2]} = -\frac{\pi \rho'(x)}{J} + \cdots \quad \text{as} \quad J \to \infty \,. \tag{2.6}$$

One can check that indeed

$$\mathcal{P} \int_{-1}^{1} \frac{(1 - y^2) dy}{(x - y)[1 + J^2(x - y)^2]} = \frac{2\pi x}{J} + \cdots \quad \text{as} \quad J \to \infty \,. \tag{2.7}$$

The key point is that this result is linear in x. From this integral, it follows that in the strong 't Hooft coupling limit, $\lambda = g^2 N \gg 1$, the solution to the integral equation (2.5) is the parabolic distribution

$$\rho(x) = \frac{3N(L^2 - x^2)}{4L^3} \,, \tag{2.8}$$

with

$$L = N^{1/2} \frac{(3\pi)^{1/3}}{2^{1/3}} \frac{1}{\lambda^{1/6}} \,. \tag{2.9}$$

The correctness of this solution can be verified by plugging it into the equation (2.5), performing the integral and then taking the large λ limit. We have also simulated the eigenvalue partition function (2.2) numerically using a Hybrid Monte-Carlo algorithm. The resulting distribution is shown for $N = 500$ and $\lambda = 600$ in figure 1. We stored 10000 configurations and determined the distribution $\rho(x)$ from the distribution of $N \times 10000$ points. The plot shows that the eigenvalue distribution is indeed close to our theoretical result (2.8). An analogous plot at lower λ (say $\lambda = 20$) shows clear deviations away from being a parabola. Note that the N appearing in these computations is just a discretisation of the integral equation (2.5). Therefore, for these purposes, we can take λ to be larger than N if we wish, without upsetting the 't Hooft limit.

We should note however, that the solution (2.8) is not correct very near to the endpoints of the distribution, $x \sim \pm L$. More specifically, it is not correct for $(L^2 - x^2) \lesssim L^2/\lambda^{1/3}$.

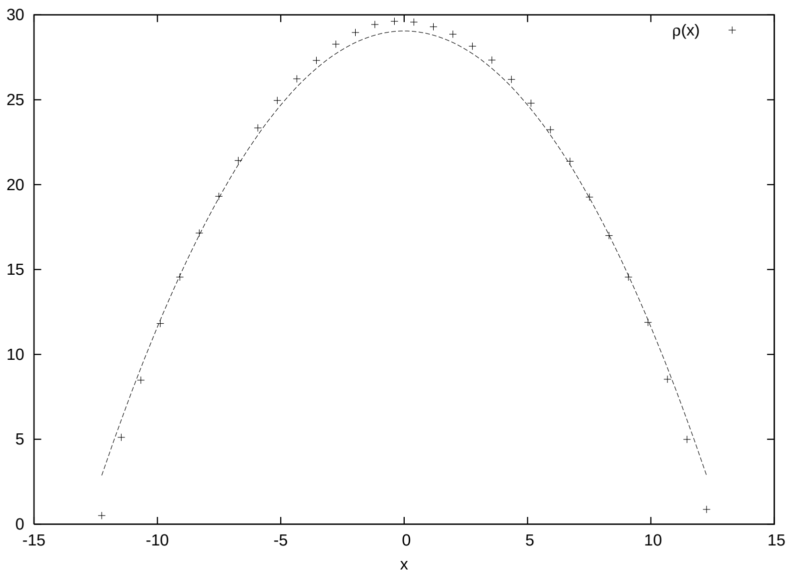

Figure 1: Numerically simulated eigenvalue distribution solution to (2.4) with $N = 500$ and $\lambda = 600$, together with the theoretical result (2.8), which has $L = 12.91$ in this case.

This occurs because there are higher order corrections to (2.7) of the form $1/[(1-x^2)J]^n$, with $n > 0$, which become large near the endpoints. However, this represents a vanishingly small fraction of the support of the eigenvalue density and so should not be important for many observables. Specifically

$$\frac{\#\text{ incorrect}}{\#\text{ total}} \sim \frac{1}{\lambda^{2/3}} \to 0\,. \tag{2.10}$$

Furthermore, by going to very large λ we have checked that the result for the width L in (2.9) agrees excellently with numerics, including the prefactor $(3\pi/2)^{1/3} \approx 1.68$.

For a typical pair of eigenvalues, the solution (2.8) implies that

$$g^2(x-y)^2 \sim \lambda L^2/N \sim \lambda^{2/3} \gg 1\,. \tag{2.11}$$

This observation suggests that the matrices X and Y are effectively commuting at strong coupling, because most of the off diagonal modes are parametrically more massive than the diagonal modes. Recall that the off diagonal mode connecting the ith and jth eigenvalue has mass $1 + g^2(x_i - x_j)^2$, as we used for instance in evaluating the determinant (2.2). We shall make this notion of commutativity more precise in the following subsection.

It is now easy to evaluate, for instance,

$$\frac{\mathrm{tr}X^2}{N} = \int_{-L}^{L} \frac{3x^2(L^2-x^2))}{4L^3}dx = \frac{L^2}{5} = \frac{(3\pi)^{2/3}N}{5\cdot 2^{2/3}\lambda^{1/3}} + \cdots\,. \tag{2.12}$$

We can note that this agrees exactly with the strong coupling result by Kazakov-Kostov-Nekrasov, see [29] equation (6.29). Thus we see explicitly that the vanishing fraction of eigenvalues near the endpoint that are incorrectly captured by a direct large λ limit do not affect this observable. By comparing with numerics we will verify that this is true for all the observables of interest in determining the emergence of a classical geometry.

2.2 Commutators and criteria for locality

Let us now consider observables that depend on both matrices X and Y in the two matrix model. Our objective here is to show more quantitatively that the two matrices commute at strong coupling. Our comment in the previous subsection about the large mass of off-diagonal modes is insufficient in itself. In the framework we are using so far, in which X is diagonalised and Y treated exactly, the off-diagonal modes do in fact make significant contributions to generic observables involving Y. We shall see this shortly. When X and Y are treated on an unequal footing, it is best to discuss basis-independent quantities. In particular, we are interested in the following combinations

$$\mathrm{tr}(XYXY)\,, \qquad \mathrm{tr}(X^2Y^2)\,, \qquad \mathrm{tr}[X,Y]^2 = 2\left(\mathrm{tr}(XYXY) - \mathrm{tr}(X^2Y^2)\right)\,. \tag{2.13}$$

Obviously the commutator square should have information on how close to commuting are two sets of matrices. The other two combinations have different large N behavior in free matrix models. The term $\mathrm{tr}(XYXY)$ would vanish at the planar level, and $\mathrm{tr}(X^2Y^2)$ would not. We would consider the case of completely uncorrelated matrices to mean that the matrices are non-commuting. The ratio

$$r = \frac{\mathrm{tr}(XYXY)}{\mathrm{tr}(X^2Y^2)}\,, \tag{2.14}$$

would then serve as an order parameter that tells us something about the correlation of the matrices. To leading order in $1/N$ it vanishes in the Gaussian model.

It is easy to show that $|r| \leq 1$ for Hermitian matrices. It results from a simple manipulation of the following two inequalities:

$$\mathrm{tr}([X,Y]^2) \leq 0\,, \tag{2.15}$$

$$\mathrm{tr}(\{X,Y\}^2) \geq 0\,. \tag{2.16}$$

These follow from the fact that $[X,Y]$ and $\{X,Y\}$ are anti-Hermitian and Hermitian respectively. For matrices that commute, we find that $r = 1$, while for matrices that anticommute we would find that $r = -1$. In the case $r = -1$, the square of the matrices would commute with each other, so there is also a lot of order in the eigenspaces of the matrices.

When we consider our two matrix model, the easiest of these to compute is the commutator squared. Specifically

$$\mathrm{tr}[X,Y]^2 = \frac{N}{Z}\frac{\partial Z}{\partial \lambda} = N\frac{\partial \log Z}{\partial \lambda}\,. \tag{2.17}$$

Here $\log Z$ should be evaluated on the large N saddle

$$\log Z = -\int \rho(x)x^2 dx + \int dxdy\rho(x)\rho(y)\left(\frac{1}{2}\log(x-y)^2 - \frac{1}{2}\log[1+g^2(x-y)^2]\right)\,. \tag{2.18}$$

Using our parabolic solution (2.8) we obtain to leading order at large λ

$$\mathrm{tr}[X,Y]^2 = -\frac{N^3}{2\lambda} + \cdots\,. \tag{2.19}$$

This is the correct answer for the commutator to leading order in large λ. It is clear from the computation that higher order corrections to the eigenvalue distribution give subleading contributions.

To get the other traces we can introduce a source J_{ij} into the action for the Y_{ij} component of Y. Let

$$Z[J] = \int \mathcal{D}X\mathcal{D}Y e^{-\mathrm{tr}X^2-\mathrm{tr}Y^2+g^2\mathrm{tr}[X,Y]^2+\mathrm{tr}JY}\,. \tag{2.20}$$

Diagonalising the X matrix, this becomes

$$Z[J] = e^{\frac{1}{4}\sum_{i\neq j}|J_{ij}|^2[1+g^2(x_i-x_j)^2]^{-1}} Z[0]\,. \tag{2.21}$$

Thus

$$\mathrm{tr}(XYXY) = \sum_{i,j} \frac{x_i x_j}{Z[0]} \frac{\delta^2 Z[J]}{\delta J_{ij}\delta J_{ji}}\bigg|_{J=0} = \int \frac{dxdy\rho(x)\rho(y)xy}{2[1+g^2(x-y)^2]}\,, \tag{2.22}$$

and similarly

$$\mathrm{tr}(X^2Y^2) = \int \frac{dxdy\rho(x)\rho(y)x^2}{2[1+g^2(x-y)^2]}\,. \tag{2.23}$$

It is easy to evaluate these integrals using our large N distribution (2.8) to obtain at large λ

$$\mathrm{tr}(XYXY) = \frac{3\pi}{70}\left(\frac{3\pi}{2}\right)^{1/3}\frac{N^3}{\lambda^{2/3}} - \frac{3N^3}{8\lambda} + \frac{\alpha N^3}{\lambda} + \cdots\,, \tag{2.24}$$

$$\mathrm{tr}(X^2Y^2) = \frac{3\pi}{70}\left(\frac{3\pi}{2}\right)^{1/3}\frac{N^3}{\lambda^{2/3}} - \frac{N^3}{8\lambda} + \frac{\alpha N^3}{\lambda} + \cdots\,. \tag{2.25}$$

In these expressions we have included an unknown contribution $\alpha N^3/\lambda$ that comes from the leading correction to the parabolic eigenvalue distribution (2.8) in the large λ limit. We know that the contribution has to be equal in the two expressions because taking their difference recovers our previous result for the commutator (2.19). Recall that our expression for the commutator did not depend on corrections to the eigenvalue distribution. The ratios we are about to consider do not depend on α. For completeness, in appendix A we show that

$$\alpha = -\frac{1}{40}\,. \tag{2.26}$$

We checked the leading order results for the commutators numerically, simulating the full partition function (2.1).

We can now ask how these observables capture the commutativity of the matrices at strong coupling. The most naïve object to look at would be

$$\frac{N\mathrm{tr}[X,Y]^2}{\mathrm{tr}X^2\mathrm{tr}Y^2} = -\frac{25}{2}\left(\frac{2}{3\pi}\right)^{4/3}\frac{1}{\lambda^{1/3}} + \cdots \to 0\,. \tag{2.27}$$

This says that the commutator is vanishing relative to observables that only depend on single matrices. The ratio cancels out the overall scaling of the X and Y matrices. Thus,

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the condition that (2.27) go to zero seems to be a natural notion of whether the matrices X and Y commute. As we discussed above, another natural ratio to consider is

$$\frac{\mathrm{tr}(XYXY)}{\mathrm{tr}(X^2Y^2)} = 1 - \frac{35}{6\pi}\left(\frac{2}{3\pi}\right)^{1/3}\frac{1}{\lambda^{1/3}} + \cdots \to 1\,. \tag{2.28}$$

Which also shows that the matrices commute. The parameter $\lambda^{1/3}$ controls the size of the non-commutativity of the matrices. If the matrices are sufficiently close to commuting, we make small errors by assuming that they are mutually diagonal, and that there is a joint eigenvalue distribution.

Therefore, we consider the behaviour of the ratios (2.27) and (2.28) as criteria for the emergence of a local geometry. Our computations above show that they are very closely related to the mass of the off diagonal modes, $m^2_{\text{o.d.}} \sim \lambda^{2/3}$, as we should expect.

One should notice that the criterion for commutativity, r can be refined further. For example, we can ask more local questions in the spectrum of X if we consider the ratios

$$r_f = \frac{\mathrm{tr}(f(X)Yf(X)Y)}{\mathrm{tr}(f(X)^2Y^2)}\,, \tag{2.29}$$

for f some real function of x which is peaked in some region. We can do the same with Y. Depending on how r_f varies with the width δ of f, we can talk of a local degree of noncommutativity on the scale of δ. It should be noted that these can be easily evaluated numerically if f is a rational function, without the need to diagonalize X. This seems to serve as a reasonable definition of the local sharpness of a fuzzy geometry.

2.3 Emergence of local geometry: hemisphere distribution

Given that most of the off diagonal modes of Y are parametrically heavy when $\lambda \gg 1$, and given the observations of the previous subsection, we might expect to be able to recast this model as a commuting matrix model for the two matrices X and Y. A two dimensional commuting matrix model is a matrix model in which the matrices are further constrained to commute. At large N the model is described by a joint eigenvalue density $\rho(x,y)$ for the eigenvalues of the commuting matrices.

It is easy to see that the hemisphere distribution

$$\rho(x,y) = \begin{cases} \dfrac{3N\sqrt{L^2-x^2-y^2}}{2\pi L^3} & \text{for} \quad x^2+y^2 < L^2 \\ 0 \quad \text{otherwise}\,, \end{cases} \tag{2.30}$$

recovers our one dimensional parabolic distribution (2.8) upon integrating out one direction

$$\rho(x) = \int_{-\sqrt{L^2-x^2}}^{\sqrt{L^2-x^2}} \rho(x,y)dy\,. \tag{2.31}$$

This immediately implies that all observables depending on only one of the matrices can be computed using this two dimensional eigenvalue distribution, which we might call the hemisphere distribution. Note that (2.30) is the unique radially symmetric distribution

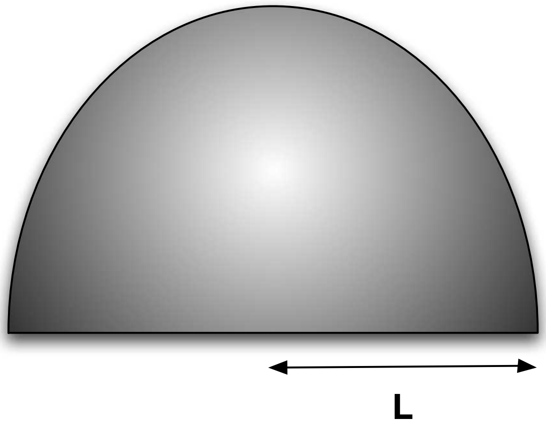

Figure 2: The emergent two dimensional hemisphere distribution.

with the property (2.31). Radial symmetry is appropriate as the original two matrix model was SO(2) invariant. The emergent hemisphere distribution is shown in figure 2.

A nontrivial test of this emergent two dimensional eigenvalue distribution is to reproduce observables that depend on both X and Y. In particular we find

$$\mathrm{tr}(XYXY) = \mathrm{tr}(X^2Y^2) = \int \rho(x,y) x^2 y^2 dx dy = \frac{3\pi}{70}\left(\frac{3\pi}{2}\right)^{1/3} \frac{N^3}{\lambda^{2/3}} \,. \tag{2.32}$$

Precisely reproducing the exact result (2.24) to leading order at large λ. Here we note that off-diagonal elements are manifestly not necessary in order to correctly compute observables involving both X and Y in this commuting framework (except for commutators, of course, which vanish to leading order). Unlike in the previous discussion in which we diagonalised only X, in this simultaneously diagonalised basis there is a well-defined separation into light eigenvalues and heavy off-diagonal modes.

It is natural to ask for an action that has the eigenvalue distribution (2.30) as its ground state. The naïve thing to do is to obtain a one loop effective action for the simultaneous eigenvalues of X and Y by integrating out the off diagonal modes. One might hope this will work even at strong coupling, because, as we saw, the off diagonal modes are becoming parametrically heavy. An estimate of when perturbation theory is valid can be obtained as follows (essentially this argument appears in, for instance, [18]). Given an eigenvalue distribution of extension L, the action for off diagonal modes δX is schematically

$$S_{\text{off-diag}} \sim (1 + g^2L^2)\delta X^2 + g^2 L \delta X^3 + g^2 \delta X^4 \,. \tag{2.33}$$

Supposing $g^2L^2 \gg 1$, so that off-diagonal modes are heavy, we can ask when the two loop contribution to the partition function is parametrically smaller than the one loop partition function. If we normalise the one loop contribution to 1, then it is easy to see that the two loop contribution is of order $g^2N/(g^2L^2)^2$. Thus the higher loop contribution is negligible if

$$L \gg \frac{N^{1/2}}{\lambda^{1/4}} \,. \tag{2.34}$$

This condition is indeed satisfied by the width of the hemisphere distribution (2.9). Emboldened, we go ahead and compute the one loop action for the simultaneous eigenvalues to obtain

$$S_{2D} = \sum_i \vec{x}_i^2 - \sum_{i \neq j} V(|\vec{x}_i - \vec{x}_j|)\,, \tag{2.35}$$

with

$$V(s) = \frac{1}{2}\log(s^2) - \frac{1}{2}\log(1+g^2 s^2)\,. \tag{2.36}$$

In integrating out the off-diagonal modes one should gauge fix, for instance as described in [23]. This is of course the same action as we obtained before, except that now we have two component vectors $\vec{x} = (x, y)$. The large N equations of motion are

$$\vec{x} = \int \frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|} V'(|\vec{x}-\vec{y}|)\rho(\vec{y}) d^2 y \tag{2.37}$$

Writing this equation out for a radially symmetric distribution

$$x = \int_0^L dr r\rho(r) \int_0^{2\pi} d\theta \frac{V'(\sqrt{x^2+r^2-2xr\cos\theta})(x - r\cos\theta)}{\sqrt{x^2+r^2-2xr\cos\theta}} \tag{2.38}$$

$$= \frac{\pi}{x}\int_0^L dr r\rho(r)\left(\frac{1+g^2(r^2-x^2)}{\sqrt{1+2g^2(r^2+x^2)+g^4(r^2-x^2)^2}} + H(x-r)\right)\,. \tag{2.39}$$

In the second step we assumed, without loss of generality, that $x > 0$. We also introduced the step function $H(s)$ which equals -1 for $s < 0$ and $+1$ otherwise.

At large gL we can solve this last equation (2.39) with some educated guesswork. The solution is

$$\rho(r) = \frac{2N}{\pi}\frac{\widetilde{L}^2 - r^2}{\widetilde{L}^4}\,. \tag{2.40}$$

where

$$\widetilde{L} = N^{1/2}\left(\frac{\log\lambda}{\lambda}\right)^{1/4}\,. \tag{2.41}$$

The correctness of this solution can be checked by performing the integral (2.39) and then taking the strong coupling limit. This clearly does not agree with the hemisphere distribution (2.30) and the width $\widetilde{L}$ has a different scaling with λ than (2.9). It is interesting to see that the eigenvalue problem (2.37) acquires a logarithmic non-analyticity in the 't Hooft coupling in the strong coupling limit. It is not easy to check the solution (2.41) numerically to high accuracy, because of the logarithmic dependence, but it easy numerically to see the distribution become paraboloid and the width scale like $1/\lambda^{1/4}$ rather than $1/\lambda^{1/6}$.

The implication of this mismatch is that the one loop effective action for the eigenvalues (2.35) is insufficient to capture the eigenvalue dynamics, despite the fact that the width (2.9) satisfies the naïve bound (2.34). We suspect that this occurs because the scaling of $L \sim 1/\lambda^{1/6}$ is fairly close to the limiting scaling $1/\lambda^{1/4}$. It may be possible to identify the required higher loop corrections to the effective action and re-obtain the correct emergent geometry. Indeed the appearance of a logarithm of the coupling suggests a resummation should be done, but we leave this for future work.

To summarise this section: we have shown that the two matrix model (2.1) becomes commuting at strong coupling and that there is an emergent two dimensional hemisphere geometry (2.30). We saw that locality of physics in this geometry, i.e. that the off diagonal modes are heavy, was captured by the observables (2.27) and (2.28). We found that describing the eigenvalue dynamics will require going beyond the one loop effective action.

3. Solvable models with more than two matrices

3.1 Bosonic model: no geometry

There is a generalisation of the two matrix model that can also be treated analytically. It is instructive to see how this model does not lead to an emergent geometry at strong coupling. Consider the $k+1$ matrix model with SO(k) symmetry only

$$Z = \int \mathcal{D}X\mathcal{D}Y_1 \cdots \mathcal{D}Y_k e^{-\mathrm{tr}X^2 - \sum_m \mathrm{tr}Y_m^2 + g^2 \sum_m \mathrm{tr}[X,Y_m]^2} . \tag{3.1}$$

This integral is again quadratic in all the Y_ms, which may therefore be integrated out exactly. The crucial simplification is the absence of interactions between the Y matrices. Diagonalising X, we obtain

$$Z = \int dx_1 \dots dx_N e^{-\sum_i x_i^2 + \frac{1}{2}\sum_{i\neq j}\log(x_i - x_j)^2 - \frac{k}{2}\sum_{i\neq j}\log[1+g^2(x_i-x_j)^2]} . \tag{3.2}$$

The large N saddle point of this integral is an eigenvalue distribution satisfying

$$x = \int \left[\frac{1-k}{x-y} + \frac{k}{(x-y)[1+g^2(x-y)^2]}\right]\rho(y)dy . \tag{3.3}$$

We have rearranged the terms a little. As before $\int \rho(x)dx = N$. The second term is a repulsive force of the same form as we found previously in (2.5). The first term, however, is now an attractive force (for $k > 1$) that is stronger at long distances. Eigenvalues separated by $(x-y)^2 = 1/(g^2(k-1))$ experience no net force. One could presumably solve this integral equation fully using the techniques in [29]. Once again, we shall look for a pedestrian approach using the strong coupling expansion.

A good starting point to attack (3.1) analytically is the limit of a large number k of matrices. The attractive force is becoming more important in this limit, so we might expect the width L of the distribution becomes small. In fact, based on the force balance equation, the natural scale to expect is $L^2 \sim 1/g^2k$. Let us assume this scaling and see where it leads.

If $L^2 \sim 1/g^2k$ and k is becoming large, then $(x-y)^2g^2 \leq 4L^2g^2$ is becoming small. Therefore we can expand the last term in (3.3). Rearranging, we obtain

$$x + kg^2 \sum_{n=0}^{\infty}(-1)^n g^{2n} \int (x-y)^{2n+1}\rho(y)dy = \int \frac{\rho(y)dy}{x-y} . \tag{3.4}$$

The first few terms in the sum read

$$(1+k\lambda)x - \frac{k\lambda^2}{N}(x^3 + 3\alpha_2 x) + \frac{k\lambda^3}{N^2}(x^5 + 10\alpha_2 x^3 + 5\alpha_4 x) + \cdots = \int \frac{\rho(y)dy}{x-y} . \tag{3.5}$$

We have used the fact that $\rho(y)$ will be even and introduced the notation for the moments of the distribution

$$\alpha_m = \frac{1}{N}\int y^m \rho(y) dy\,. \tag{3.6}$$

Note that each term in this expansion is suppressed by a factor of $L^2 g^2 \sim 1/k$ compared to the previous terms. Therefore we can solve order by order in the large k limit. The leading order solution is clearly the semicircle

$$\rho(x) = \frac{2N}{\pi L^2}\sqrt{L^2 - x^2}\,, \tag{3.7}$$

with

$$L = \sqrt{\frac{2N}{1+k\lambda}}\,. \tag{3.8}$$

This expression implies $L^2 g^2 \ll 1$ for all λ, in the large k limit. Therefore this is the correct leading order solution at large k for all couplings, not just strong coupling. It is not surprising that upon integrating out a large number of matrices one obtains the semicircle distribution, corresponding to effectively Gaussian degrees of freedom. We have checked that (3.8) agrees excellently with numerical results.

To second order, the solution may be found using standard matrix model techniques. It is straightforward to verify that the solution to second order is

$$\rho(x) = \frac{N}{\pi L^2}\sqrt{L^2 - x^2}\left(2 + \frac{k\lambda^2 L^4}{4N^2} - \frac{k\lambda^2 L^2}{N^2}x^2\right)\,, \tag{3.9}$$

where the width L satisfies (to this order in large k)

$$4N^2 - 2NL^2(1+k\lambda) + 3kL^4\lambda^2 = 0\,. \tag{3.10}$$

If we work in the large λ limit for simplicity, this is seen to imply a corrected eigenvalue width of

$$L = \sqrt{\frac{2N}{k\lambda}}\left(1 + \frac{3}{2k}\right)\,. \tag{3.11}$$

We checked these results by simulating the eigenvalue partition function (3.2) numerically. We again used Hybrid Monte-Carlo with $N = 100$ and stored 1000 configurations, determining the distribution from $N \times 1000$ points. The resulting distribution for $\lambda = 1000$ and $k = 20$ is shown in figure 3 together with the theoretical expectation (3.9). There is an excellent agreement.

It is straightforward to move systematically to arbitrary order. For instance the seventh order solution is of the form

$$\rho(x) = \frac{N}{\pi}\sqrt{L^2 - x^2}\left(a + bx^2 + cx^4 + dx^6 + ex^8 + fx^{10} + gx^{12}\right)\,, \tag{3.12}$$

with the constants $a, \ldots, g$ determined by plugging this expression into (3.4). The width of the distribution (at large λ) is found to be at this order

$$L = \sqrt{\frac{2N}{\lambda k}}\left(1 + \frac{3}{2}\frac{1}{k} + \frac{5}{4}\frac{1}{k^2} + \frac{11}{8}\frac{1}{k^3} + \frac{57}{32}\frac{1}{k^4} + \frac{33}{64}\frac{1}{k^5} + \frac{39}{16}\frac{1}{k^6}\right)\,. \tag{3.13}$$

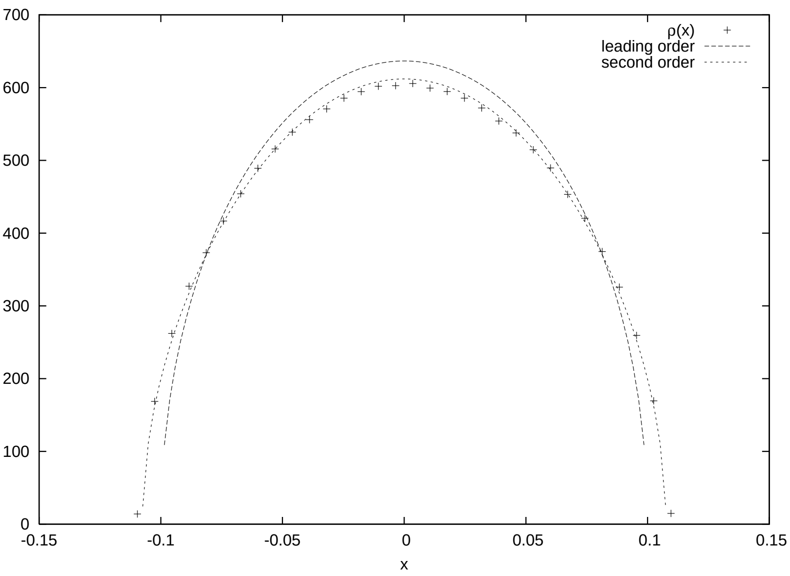

Figure 3: Numerically simulated eigenvalue distribution solution to (3.2) with $N = 100$, $k = 20$ and $\lambda = 1000$, together with the theoretical result to first (3.7) and second (3.9) order in a $1/k$ expansion. The leading order distribution has $L = 0.0975$ and the subleading distribution $L = 0.1075$.

For $k = 2$ this formula gives $L = 2.3999\sqrt{N/\lambda}$ with small corrections. Given that $k = 2$ is the smallest case we have to consider, this implies that (3.12) and (3.13) are a good approximation to the solution for all integer $k > 1$. The solution has $L^2 \sim 1/g^2 k$, and so selfconsistently satisfies our scaling assumption. It seems plausible that the radius of convergence of the series (3.13) will be $k = 1$, consistent with the fact that we found a different scaling with λ in that case.

The implication of the above results, e.g. (3.13), is that for these multi-matrix models, with $k \geq 2$, the mass of the off diagonal modes of the matrix X is of the same order as the diagonal modes. This is because $L^2 g^2$ is order one or smaller. Therefore we do not expect an effective description in terms of the simultaneous eigenvalues of X and Y. Furthermore, we don't expect the Ys to commute amongst themselves as they only couple to one another through X. Let us see if these expectations are reflected in the observables we considered previously.

For simplicity, let us work to leading order at large λ and k. That is, we take the solution (3.7) for $\rho(x)$. We have seen that the scaling of quantities with λ does not change away from this limit down to $k = 2$.

The two point function of X is easy to evaluate

$$\frac{\mathrm{tr}X^2}{N} = \frac{1}{N}\int_{-L}^{L} x^2 \rho(x) dx = \frac{L^2}{4} = \frac{N}{2k\lambda} \,. \tag{3.14}$$

To compute correlation functions involving the Ys, we introduce source terms as before

$$Z[J] = \int \mathcal{D}X\mathcal{D}Y_1 \cdots \mathcal{D}Y_k e^{-\mathrm{tr}X^2 - \sum_m \mathrm{tr}Y_m^2 + g^2 \sum_m \mathrm{tr}[X,Y_m]^2 + \sum_m \mathrm{tr}J_m Y_m} \,. \tag{3.15}$$

Using exactly the same steps as in section 2.3 above, we can obtain the two point function for the Ys

$$\frac{\mathrm{tr}Y_n^2}{N} = \frac{1}{N}\int \frac{\rho(x)\rho(y)dxdy}{2(1+g^2(x-y)^2)} \approx \frac{1}{2N}\int \rho(x)\rho(y)dxdy = \frac{N}{2}\,. \tag{3.16}$$

In this and subsequent formulae we are using the fact noted above that $(x-y)^2g^2 \leq 4L^2g^2 \ll 1$ for this distribution. We see that the Ys are spread out by an extra factor of λ compared to the X matrix. The four point functions $\mathrm{tr}X^2Y^2$ and $\mathrm{tr}XYXY$ are

$$\begin{aligned}\mathrm{tr}X^2Y_n^2 &= \int \frac{\rho(x)\rho(y)x^2dxdy}{2(1+g^2(x-y)^2)} \\ &\approx \frac{1}{2}\int \rho(x)\rho(y)x^2dxdy = \frac{L^2N^2}{8} = \frac{N^3}{4k\lambda}\,,\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\mathrm{tr}XY_nXY_n &= \int \frac{\rho(x)\rho(y)xydxdy}{2(1+g^2(x-y)^2)} \\ &\approx -\frac{g^2}{2}\int (x-y)^2xy\rho(x)\rho(y)dxdy = \frac{\lambda L^4 N}{16} = \frac{N^3}{4k\lambda}\frac{1}{k}\,.\end{aligned} \tag{3.18}$$

This second expression is suppressed by an extra factor of $1/k$ and so can be neglected to leading order at large k. It follows that to leading order $\mathrm{tr}[X,Y]^2 = -2\mathrm{tr}X^2Y^2$.

The four point functions involving two different Ys are

$$\mathrm{tr}Y_m^2Y_n^2 = \int \frac{\rho(x)\rho(y)\rho(z)dxdydz}{4(1+g^2(x-z)^2)(1+g^2(y-z)^2)} \approx \frac{1}{4}\int \rho(x)\rho(y)\rho(z)dxdydz = \frac{N^3}{4}\,, \tag{3.19}$$

and

$$\mathrm{tr}Y_mY_nY_mY_n = \frac{N}{4} \sim 0\,. \tag{3.20}$$

This last term is subleading in $1/N$ and therefore effectively zero in the planar approximation we are taking. The following four point functions are also obviously zero at large N, from the absence of Y mixing in the classical action

$$\mathrm{tr}XY_mXY_n = \mathrm{tr}X^2Y_mY_n = 0\,. \tag{3.21}$$

We have checked all the four point functions in this section by comparing with numerical results obtained by simulating the full partition function (3.1).

We can now compute our ratios to be

$$\frac{N\mathrm{tr}[X,Y_n]^2}{\mathrm{tr}X^2\mathrm{tr}Y_n^2} = \frac{N\mathrm{tr}[Y_m,Y_n]^2}{\mathrm{tr}Y_m^2\mathrm{tr}Y_n^2} = -2\,, \tag{3.22}$$

and

$$\frac{\mathrm{tr}XY_nXY_n}{\mathrm{tr}X^2Y_n^2} = \frac{1}{k} \to 0\,. \tag{3.23}$$

Both of these expressions are consistent with our expectation that this model is not commuting. The fact that the first ratio is not going to zero and the second is less than one

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will remain true at finite k. It is interesting to note that at finite k the second ratio is not driven to zero as the coupling goes to infinity. Therefore the finite k models are not 'maximally non-commuting'. For the Ys however, we do have at large N that

$$\frac{\mathrm{tr}Y_m Y_n Y_m Y_n}{\mathrm{tr}Y_m^2 Y_n^2} = 0\,. \tag{3.24}$$

A simple but important lesson to draw from this model is that quantum loop effects are crucial in determining whether the strongly coupled system is commuting or not. Here, the extra one loop contributions from the matrices destabilised the emergent geometry of the two matrix model. This suggests that we can improve the situation by adding fermionic matrices to cancel the undesired loop contributions.

A final point to note, from e.g. (3.5), is that the mass term is unimportant in determining the strong coupling eigenvalue distribution in this model. This is in contrast to the (commuting at strong coupling) two matrix case we considered previously. Without a mass term, the coupling λ can simply be absorbed into the normalisation of the X matrix and therefore does not have any dynamics associated to it. In particular, strong coupling cannot drive us to commutativity in a massless theory. A lesson to draw, therefore, is that commutativity should involve an interplay been the mass term and the 'commutator square' interaction terms.

3.2 Model with fermionic matrices: still no geometry

It has long been appreciated that supersymmetry facilitates the emergence of geometry [30–32]. Here we show how a simple implementation of this idea works for us. However, the absence of direct interactions between the Y matrices will ultimately prevent the emergence of a geometry involving all the matrices in these solvable models.

We can supplement the $1+k$ matrix model (3.1) with $2h$ fermionic fields as follows

$$Z = \int \mathcal{D}X\mathcal{D}Y_1\cdots\mathcal{D}Y_k\mathcal{D}\lambda_1\cdots\mathcal{D}\lambda_h\mathcal{D}\mu_1\cdots\mathcal{D}\mu_h \\ \times e^{-\mathrm{tr}X^2-\sum_m \mathrm{tr}Y_m^2-\sum_n \mathrm{tr}\lambda_n\mu_n+g^2\sum_m \mathrm{tr}[X,Y_m]^2+ig\sum_n \mathrm{tr}\lambda_n[X,\mu_n]}\,. \tag{3.25}$$

In this model, the λ_n and μ_n are Hermitian N by N matrices of anticommuting numbers. Each matrix component of λ and μ has only a single component, no 'spinor index' is necessary. Because of the asymmetry of the interactions, we are able to add fermions without introducing a Clifford algebra and without breaking any of the symmetries of the bosonic action.

The partition function is quadratic in all the Ys and fermions, which we integrate out exactly to give

$$Z = \int dx_1\ldots dx_N e^{-\sum_i x_i^2+\frac{1}{2}\sum_{i\neq j}\log(x_i-x_j)^2+\frac{h-k}{2}\sum_{i\neq j}\log[1+g^2(x_i-x_j)^2]}\,. \tag{3.26}$$

We see that including the fermions can cancel out the attractive part of the potential, thus increasing the likelihood that the eigenvalues of X will spread out sufficiently to give the off diagonal modes with the Ys a large mass. We already know the answer for the eigenvalue distribution in several cases:

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- If $h < k-1$, the partition function is (3.1) with $k' = k-h > 1$ and one obtains the same results as in the previous subsection, with no emergent geometry.
- If $h = k-1$, the partition function becomes precisely that of section 2, leading to a parabolic distribution of the eigenvalues of X with width $L \sim g^{-1/3}$ at strong coupling.
- For the case $h = k$, the partition function is that for a Gaussian model and clearly results in a semi-circle distribution for the eigenvalues of X with width $L \sim N^{1/2}$.

In the latter two cases, $g^2 L^2$ becomes large at strong coupling and therefore the off diagonal modes between X and any of the Ys become heavy. Thus they are candidates for emergent geometry. However, we will shortly see that the Ys will not commute amongst themselves. This is essentially an artifact of the simple action we have taken, with no interactions between the Ys. Generically, if two Hermitian matrices Y_m and Y_n both commute with X then they should commute amongst themselves. The exception is if X has degenerate eigenvalues. This is effectively what is happening here.

For the remaining cases, with $h > k$, all the terms in the effective action lead to repulsive forces. It is useful to write the large N equations of motion as

$$x = \int \left[\frac{1+h-k}{x-y} + \frac{k-h}{(x-y)[1+g^2(x-y)^2]} \right] \rho(y) dy \,. \tag{3.27}$$

Because the force is more repulsive than the case $h = k$, we expect that the distribution $\rho(x)$ will have width $L \gtrsim N^{1/2}$. This would imply that $gL \gg 1$ at strong coupling. Assuming this, then from (2.6) we can ignore the last term in (3.27). It follows that the distribution at strong coupling is a semicircle

$$\rho(x) = \frac{2N}{\pi L^2} \sqrt{L^2 - x^2} \,, \tag{3.28}$$

with width

$$L = \sqrt{2N(1+h-k)} \,. \tag{3.29}$$

This result includes the case $h = k$. The result self-consistently satisfies $gL \gg 1$ in the strong 't Hooft coupling limit. Therefore we can conclude that for all $h \geq k-1$, the off diagonal modes of the Y matrices relative to X become heavy.

It follows that

$$\frac{\mathrm{tr} X^2}{N} = \frac{L^2}{4} = \frac{N(1+h-k)}{2} \,, \tag{3.30}$$

and, to leading order at large λ using for instance (2.6) to compute the large λ limit,

$$\frac{\mathrm{tr} Y_n^2}{N} = \frac{8N}{3\pi g L} = \frac{4\sqrt{2}N}{3\pi(1+h-k)^{1/2}} \frac{1}{\lambda^{1/2}} \,. \tag{3.31}$$

The commutator can be computed robustly as we did for the two matrix case, by differentiating the logarithm of the partition function with respect to λ. To pick out the bosonic commutator squared, one should take the couplings in front of the bosonic and fermionic interactions to be distinct before differentiating. We find

$$\mathrm{tr}[X,Y_n]^2 = -\frac{N^3}{2\lambda}\,. \tag{3.32}$$

Also proceeding as previously, we find the four point functions

$$\begin{aligned}
\mathrm{tr}X^2Y_n^2 &= \frac{8LN^2}{15\pi g} - \frac{N^2}{4g^2} + \frac{N^2\beta}{g^2} \\
&= \frac{8\sqrt{2}(1+h-k)^{1/2}N^3}{15\pi}\frac{1}{\lambda^{1/2}} - \frac{N^3}{4\lambda} + \frac{N^3\beta}{\lambda}\,, && (3.33)\\
\mathrm{tr}XY_nXY_n &= \frac{8LN^2}{15\pi g} - \frac{N^2}{2g^2} + \frac{N^2\beta}{g^2} \\
&= \frac{8\sqrt{2}(1+h-k)^{1/2}N^3}{15\pi}\frac{1}{\lambda^{1/2}} - \frac{N^3}{2\lambda} + \frac{N^3\beta}{\lambda}\,, && (3.34)\\
\mathrm{tr}Y_m^2Y_n^2 &= \frac{3N^3}{4g^2L^2} = \frac{3N^3}{8(1+h-k)}\frac{1}{\lambda}\,, && (3.35)\\
\mathrm{tr}Y_mY_nY_mY_n &= \frac{N}{4} \sim 0\,. && (3.36)
\end{aligned}$$

As before, there is an unknown subleading contribution $N^3\beta/\lambda$ due to the leading correction to the eigenvalue distribution at large λ. We know that the contribution is equal in both the expressions above, because the difference has to reproduce the commutator (3.32). We do not need the value of β for our computations below, and we do not attempt to calculate it.

The above results hold for $h \geq k$. In the 'critical' case $h = k-1$, the four point correlator of the Ys was not computed previously. Using the distribution (2.8) we find to leading order at strong coupling

$$\mathrm{tr}Y_m^2Y_n^2 = \frac{27\pi^2N^3}{280g^2L^2} = \frac{9\pi N^3}{140}\left(\frac{3\pi}{2}\right)^{1/3}\frac{1}{\lambda^{2/3}}\,. \tag{3.37}$$

We can now discuss the emergence of geometry in the models with $h \geq k-1$, that is, with sufficiently many fermions that the off diagonal modes of X with any of the Y_m matrices are heavy at strong coupling $\lambda \to \infty$. Two comments apply to all of these cases. Firstly, as we should expect, the X matrix commutes with the Y_m matrices according to both of our criteria. Namely

$$\frac{N\mathrm{tr}[X,Y_m]^2}{\mathrm{tr}X^2\mathrm{tr}Y_m^2} \to 0\,, \qquad \frac{\mathrm{tr}XY_mXY_m}{\mathrm{tr}X^2Y_m^2} \to 1\,, \qquad \text{as} \qquad \lambda\to\infty\,. \tag{3.38}$$

This is easily seen from our above expressions for the relevant two and four point functions.

Secondly, although one might therefore have expected the Ys to commute amongst themselves, this is not the case. For all of these models it follows from our above expressions that

$$\frac{N\mathrm{tr}[Y_m,Y_n]^2}{\mathrm{tr}Y_m^2\mathrm{tr}Y_n^2} \to \mathcal{O}(1)\,, \qquad \frac{\mathrm{tr}Y_mY_nY_mY_n}{\mathrm{tr}Y_m^2Y_n^2} \sim 0\,, \qquad \text{as} \qquad \lambda\to\infty\,. \tag{3.39}$$

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The fact that the Ys do not commute amongst themselves can be understood as being due to the absence of interactions between the Y matrices in the action of (3.25). At leading order in large N they are simply uncorrelated and cannot commute. Unfortunately, the very simplification that allowed us to get an analytic handle on this model undoes the possibility of an emergent geometry in which all of the matrices participate. The lesson we might take away is that genericity is another important property in the search for commutating models: if a matrix has degenerate eigenvalues it can commute with two other matrices without the other matrices needing to commute amongst themselves.

Given that the X matrix commutes with all of the Y matrices, one might imagine picking one of the Y matrices, say Y_1 (or perhaps some combination of them, to preserve the $\mathrm{SO}(k)$ invariance) and considering a geometry given by the joint eigenvalue distribution of this matrix with X. However, this geometry is not especially useful, even in the critical case $h = k - 1$, where the X and Y_1 eigenvalues would have roughly the same spread.[2] The problem is that the physics of the remaining Y matrices would not be local in the y_1 direction. Off diagonal modes of Y_2 (say) connecting different values of y_1 would have a comparable effect on the dynamics as the eigenvalues of Y_2. So the strongly coupled matrix model is not solved by local two dimensional geometric physics in these cases. For emergent local geometry, *all* modes relating far away spacetime points should be massive compared to local modes.

The natural next step, in search for higher dimensional geometry, is to introduce interactions between the Ys. This substantially increases the difficulty of solving the model analytically.

4. The fully interacting multi-matrix model

4.1 Bosonic model

The interacting p-matrix bosonic model with full $\mathrm{SO}(p)$ invariance is

$$Z = \int \mathcal{D}X_1 \cdots \mathcal{D}X_p e^{-\sum_m \mathrm{tr} X_m^2 + \frac{1}{2} g^2 \sum_{m,n} \mathrm{tr}[X_m, X_n]^2} . \tag{4.1}$$

We cannot solve this model exactly, for $p > 2$, even in a strong coupling expansion. However, building on the intuition from previous cases, a few observations are possible.

We do not expect this bosonic model to be commuting. We saw above that multiple one loop bosonic contributions result in an attractive potential at long distances. The eigenvalues were therefore insufficiently spread out for the off-diagonal modes to become parametrically heavier than the eigenvalues at strong coupling. Thus we do not expect an emergent geometry. We will study a commuting ansatz for a similar matrix model shortly. Let us suppose for the moment that all the elements of the matrices, diagonal and off diagonal, are of the same order. By $\mathrm{SO}(p)$ invariance, the entries will be of the same

[2]The cases with $h \geq k$ are even worse. The spread of the Y_1 eigenvalues is smaller by a power of λ compared to the spread of the X eigenvalues, see (3.30) and (3.31). In commuting matrix models, a large anisotropy can prevent the smaller dimension from emerging at all [33, 34].

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magnitude for all the matrices

$$\langle X_{ij} \rangle \sim \frac{1}{m_{\text{eff.}}} \,. \tag{4.2}$$

Here $m_{\text{eff.}}$ is the effective mass of the matrix elements. If the matrices become heavy at strong 't Hooft coupling, then we might hope to trust a one loop evaluation of their masses

$$m_{\text{eff.}}^2 \sim 1 + g^2 \sum_k \langle X_{ik} X_{kj} \rangle \sim \frac{\lambda}{m_{\text{eff.}}^2} \,. \tag{4.3}$$

From which we would conclude that the typical matrix entry scales as

$$\langle X_{ij} \rangle \sim \frac{1}{\lambda^{1/4}} \,. \tag{4.4}$$

Note that if we diagonalise one of the X matrices, with entries of order (4.4), we will obtain diagonal elements of order $N^{1/2}/\lambda^{1/4}$. This is seen, for instance, from the observation that (4.4) implies $\text{tr}X^2 = X_{ij}X_{ji} \sim N^2/\lambda^{1/2}$ whereas for a diagonal matrix $\text{tr}X^2 = X_{ii}^2$. Thus (4.4) is precisely on the boundary of our condition (2.34) for trusting a one loop effective action. We will shortly give an independent argument supporting the scaling (4.4).

In the previous paragraph, using selfconsistency and a non-commutativity assumption, we obtained a scaling for the spread of matrix elements with λ at strong coupling. The SO(p) invariance of the model was also important; in section 3.1 above an anisotropic model gave different scalings.

A consistency check of the above picture is that we can compute one of our ratios, following e.g. [18]. Consider the change of variables $X_m \to (1+\epsilon)X_m$. This must leave the partition function invariant. To linear order the measure for each matrix changes as $\mathcal{D}X_m \to (1 + (N^2-1)\epsilon)\mathcal{D}X_m$. Requiring the partition function to be invariant and using the SO(p) symmetry leads to

$$p(N^2-1) = 2p\langle \text{tr}X_m^2 \rangle - 2g^2 p(p-1)\langle \text{tr}[X_m, X_n]^2 \rangle \,. \tag{4.5}$$

This is an exact expression. Now using the scaling (4.4) at large coupling and taking the large N limit we find

$$\frac{N\text{tr}[X_m, X_n]^2}{\text{tr}X_m^2 \text{tr}X_n^2} = \frac{-N^3}{2(p-1)g^2(\text{tr}X_m^2)^2} \sim \mathcal{O}(1) \,. \tag{4.6}$$

Thus, consistently, the matrices are indeed not commuting.

At this point we can note a general result for models with these types of interactions. The model will be commuting, according to the observable (4.6), if and only if the eigenvalues are more spread out than $N^{1/2}\lambda^{-1/4}$ as $\lambda \to \infty$. Indeed, so far the only model we have discussed that satisfied this property for all the matrices involved was the two matrix model. The property is precisely the same condition that we found for the one loop effective action to be reliable. Although, as we saw in the two matrix case, this latter condition is not precise.

Consistently with our discussion at the end of section 3.1 above for non-commuting models, we see that the mass term is not playing a role in any of our strong coupling considerations. See for instance (4.3) or (4.5). This allows us to make contact with previous numerical and analytic results on the model (4.1) in the absence of a mass term [15, 16, 18, 35].

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If we naïvely take the strong coupling limit of the action (4.1), then we might expect to be able to drop the mass term. In the massless theory we could eliminate the coupling by rescaling the matrices $X \to X\lambda^{-1/4}$. It is then immediate that the entries of X will scale like $X_{ij} \sim \lambda^{-1/4}$ and therefore that eigenvalues will scale like $x \sim N^{1/2}\lambda^{-1/4}$. This scaling was indeed found numerically in [18] for $p > 2$. In [35] it was proven that bosonic matrix integrals without a mass term are convergent at sufficiently large N for $p > 2$. These results support our discussion above, including the observation that the mass term appears to indeed be unimportant at strong coupling in these models. For $p = 2$ the mass term is important because there is an infrared divergence in the massless model [15, 16] due to the zero modes in the commutator squared potential. This is consistent with the fact that we found a different scaling in the two matrix model case.

The bottom line for these fully interacting bosonic models with $p > 2$ appears to be that the strong coupling physics essentially reduces us to the massless case. The scaling of quantities with λ is fixed by dimensional analysis and the model is not commuting. Diagonal and off-diagonal modes contribute equally to generic observables, so there is no emergent local geometry. Clearly the arguments in this subsection are not intended to be as rigorous as our previous considerations.

4.2 'Supersymmetrised' models

We found in a solvable model above that adding fermions to the model such that the fermion determinant cancelled the bosonic one loop contribution allowed the eigenvalues to spread out. Supersymmetrisations of the massless fully interacting bosonic model exist in dimensions $p = 3, 4, 6, 10$, corresponding to $\mathcal{N} = 2, 4, 8, 16$ real supercharges, respectively. These have been studied numerically in for instance [16, 19] and analytically in [36]. In the cases of $p = 6$ and $p = 10$, the sign problem of the fermion determinants means that thinking in terms of a positive eigenvalue distribution is potentially misleading. However, in the cases of $p = 3$ and $p = 4$ there is no sign problem at large N [14, 19]. Furthermore, the (massless) cases $p = 3$ and $p = 4$ show strong infrared effects: the partition function is divergent for $p = 3$ whereas for $p = 4$ the partition function is finite but all moments $\langle \mathrm{tr} X^{2m} \rangle$ diverge [16, 36]. Therefore these models are excellent candidates for an emergent commuting geometry upon adding a mass term, as we would expect a balance between the repulsive interactions and the confining mass term.

Adding a supersymmetric mass term to the massless models is not straightforward, however. Instead we will consider simpler models. We start with the following SO(p) invariant matrix model, defined for all p, with $2(p-1)$ Hermitian fermionic matrices λ_m and μ_m

$$Z = \int \mathcal{D}X_1 \cdots \mathcal{D}X_p \mathcal{D}\lambda_1 \cdots \mathcal{D}\lambda_{p-1} \mathcal{D}\mu_1 \cdots \mathcal{D}\mu_{p-1} \times e^{-\sum_m \mathrm{tr} X_m^2 + \frac{1}{2} g^2 \sum_{m,n} \mathrm{tr}[X_m, X_n]^2 - \sum_m \lambda_m \sqrt{1 + \frac{1}{2} g^2 \sum_n [X_n, \bullet]^2}\, \mu_m} . \tag{4.7}$$

The fermionic interaction has been chosen so that the bosonic and fermionic one loop determinants cancel. Note that $2(p-1)$ is the appropriate number of massive fermions to cancel the massive bosons, because $X_m \to U X_m U^{-1}$ is a massless mode. In integrating out the off diagonal modes, one should for instance gauge fix as described in [23, 27].

Let us assume that this model is commuting at strong coupling and see if we can find a self consistent solution. If the off diagonal modes are heavy about this commuting background, we can hope that a one loop integration is sufficient. The eigenvalues are now vectors in $\mathbb{R}^p$ given by $\vec{x}_i = (x_i^1, \ldots, x_i^p)$. The partition function becomes

$$Z = \int d\vec{x}_1 \ldots d\vec{x}_N e^{-\sum_i \vec{x}_i^2 + \sum_{i \neq j} \frac{1}{2} \log |\vec{x}_i - \vec{x}_j|^2} . \tag{4.8}$$

The $\log |\vec{x}_i - \vec{x}_j|^2$ term can be thought of as a generalised Vandermonde determinant arising from simultaneously diagonalising the p matrices. The large N equations of motion are

$$\vec{x} = \int d^p y \rho(\vec{y}) \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^2} . \tag{4.9}$$

These equations of motion have been considered in some detail in, for instance, [23, 27, 34]. The solution is given by a $(p-1)$ sphere with constant eigenvalue density

$$\rho(\vec{y}) = \frac{N}{|\vec{y}|^{p-1} \mathrm{Vol} S^{p-1}} \delta \left(|\vec{y}|^2 - \frac{N}{2} \right) . \tag{4.10}$$

From the spherical eigenvalue distribution (4.10) one immediately obtains

$$\frac{\mathrm{tr} X_m^2}{N} = \frac{N}{2p} . \tag{4.11}$$

At strong coupling, the width of the eigenvalue distribution (4.10) is well within our bound (2.34) for when the one loop effective potential is expected to be valid. Therefore it seems likely that we have found a consistent saddle point for the full model. Without having solved the model exactly we cannot prove that this commuting saddle is the dominant large N saddle of the integral. However, we can see that (4.9) describes a balance between a mass term and an eigenvalue repulsion of the type we found previously in a commuting model. The eigenvalues have spread out as far as the mass term allows and simultaneously made the off diagonal modes parametrically heavy at strong coupling. Therefore, there would seem to be a good chance that this model is indeed showing an emergent geometry at strong coupling.

The eigenvalue distribution (4.10) together with the validity of the one loop effective action allows us to compute the commutator

$$\begin{aligned} \mathrm{tr}[X_m, X_n]^2 &= -\frac{1}{p} \int \frac{|\vec{y} - \vec{y}'|^2 \rho(\vec{y}) \rho(\vec{y}') d^p y d^p y'}{1 + g^2 |\vec{y} - \vec{y}'|^2} \\ &= -\frac{N^3 \mathrm{Vol} S^{p-2}}{p \mathrm{Vol} S^{p-1}} \int_0^\pi \frac{(\sin\theta)^{p-2} (1 - \cos\theta)}{1 + \lambda(1 - \cos\theta)} d\theta \\ &\to -\frac{N^3}{p} \frac{1}{\lambda} , \qquad \text{as} \qquad \lambda \to \infty . \end{aligned} \tag{4.12}$$

Here we used the fact that the sphere has radius $r^2 = N/2$. We can also compute the leading order four point functions

$$\mathrm{tr} X_m^2 X_n^2 = \mathrm{tr} X_m X_n X_m X_n = \int d^p y \rho(\vec{y}) y_m^2 y_n^2 = \frac{r^4}{p(p-1)} - \frac{\langle y_m^4 \rangle_{S^{p-1}}}{p-1} = \frac{N^3}{4p(p+2)} . \tag{4.13}$$

This computation simply involves integrations over the $(p-1)$ sphere. It should be possible to test these commuting saddle results using a numerical simulation of the full partition function (4.7).

Other models with SO(p) invariance that have an a priori chance of having commuting saddles are the massless supersymmetric models we described at the start of this section together with a mass term for the bosons only:

$$Z = \int \mathcal{D}X\mathcal{D}\Psi e^{-\sum_m \mathrm{tr}X_m^2 + \frac{1}{2}g^2\sum_{m,n}\mathrm{tr}[X_m,X_n]^2 + g\sum_{m,\alpha,\beta}\mathrm{tr}\Psi_\alpha[\Gamma^m_{\alpha\beta}X_m,\Psi_\beta]} \,. \qquad (4.14)$$

We can use the gamma matrix conventions of for instance [14]. In any case all that is important for us is that in the massless case the one loop bosonic and fermionic determinants cancel about a commuting background. We ignore for the moment the fermion zero modes. Therefore in our model, the one loop effective action about a commuting background is

$$Z = \int d\vec{x}_1 \dots d\vec{x}_N e^{-\sum_i \vec{x}_i^2 + \frac{p-1}{2}\sum_{i\neq j}\left[\log|\vec{x}_i - \vec{x}_j|^2 - \log(1+g^2|\vec{x}_i-\vec{x}_j|^2)\right]} \,. \qquad (4.15)$$

The equation of motion for a radially symmetric eigenvalue distribution is therefore ($p \geq 3$)

$$x = (p-1)\mathrm{Vol}S^{p-2}\int_0^L dr\rho(r)r^{p-1}\int_0^\pi d\theta\frac{\sin^{p-2}\theta V'(\sqrt{x^2+r^2-2xr\cos\theta})(x-r\cos\theta)}{\sqrt{x^2+r^2-2xr\cos\theta}} \,. \qquad (4.16)$$

In this expression the potential V is precisely as in (2.36) above. As in section 2.3 above, we can analytically perform the angular integral in (4.16) for any given p.

We have not found an analytic solution for $\rho(r)$ in (4.16). It is straightforward to see that if one inserts a generic ansatz for $\rho(r)$ into the integral equation then the width of the corresponding distribution will be $L \sim 1/\lambda^{1/4}$. This is on the borderline for selfconsistency of the one-loop action. Furthermore, we have simulated the commuting matrix model (4.15) to rather high accuracy in N and λ. The results for the width of the eigenvalue distribution are shown in figure 4 for the cases of most interest, $p = 3$ and $p = 4$.

The results of figure 4 show that the width of the eigenvalue distribution at strong coupling is a little greater than $L \sim 1/\lambda^{1/4}$. This is reminiscent of what we found previously for the commuting bosonic two matrix model in section 2.3, where the width was enhanced by a $\log\lambda$ factor. One possibility is therefore that these models are indeed commuting and that the eigenvalue dynamics is not captured by the one loop action (4.15). It would be interesting to test this possibility by simulating the full partition function (4.14) numerically and computing our observables (2.27) and (2.28). We cannot say much beyond this for this model with the results we have here.

A curious observation we can make is that if we combine the two plots of figure 4 into one plot, then the two lines essentially lie on top of each other.

One new feature that arises in this last model is that there are fermion zero modes about the commuting saddle. We did not add a mass term for the fermions and therefore fermionic matrices that simultaneously commute with the bosonic matrices have zero action. Therefore the partition function will vanish unless we insert some fermions to mop up the zero modes. Specifically we should consider an insertion like

$$\langle \prod_\alpha \Psi^\alpha_{11}\cdots\Psi^\alpha_{NN}\rangle \,. \qquad (4.17)$$

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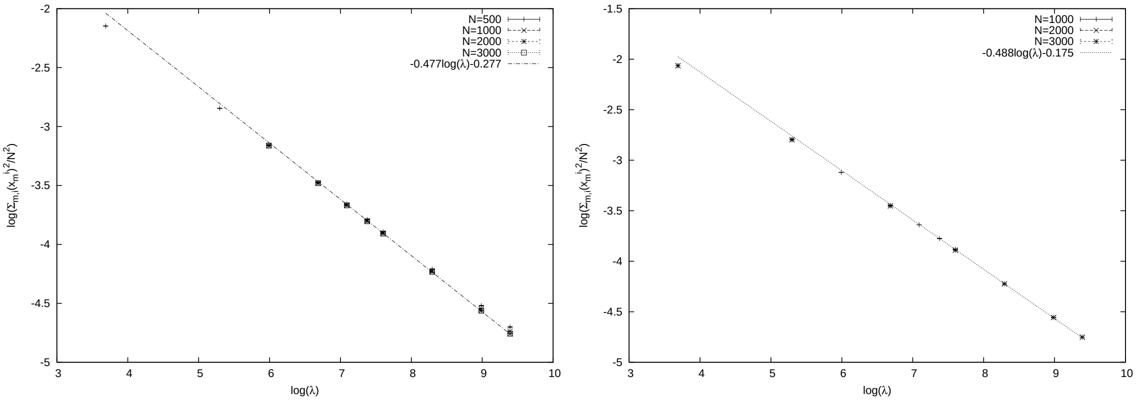

Figure 4: For the $p = 3$ model (left) an the $p = 4$ model (right), plots of $\log(\lambda)$ against $\log(\sum_m \sum_{i=1}^N (x_m^i)^2/N^2)$. The fitting line is obtained from last two points, $\lambda = 8000$ and $\lambda = 12000$, with $N = 3000$.

We have not written this observable in an explicitly basis independent way, but rather in the basis given by the commuting Xs. The Ψ matrices are anticommuting, so the observable has to be antisymmetrisable in the components of Ψ, as well as having N entries for each matrix.

5. Classical fuzzy spheres

There is another notion of geometry that appears repeatedly in studies of matrix theory and string theory. These are non-commutative spaces formed by the classical matrix degrees of freedom of collections of D branes. For instance, the D0 brane matrices can form a regularized version of a higher dimensional membrane geometry (this was a crucial insight to develop matrix theory [30]). Such blowing up into higher dimensions often occurs due to the Myers effect [37], and it can also show up in the classification of vacua of some supersymmetric field theories [38].

A typical example of such spaces are fuzzy spheres, whose classical coordinates satisfy the SU(2) Lie algebra relations. There are many other such configurations that have been studied. In this section we will make a small study of fuzzy spheres, to complement our points of view from the rest of the paper.

The first thing we would like is a matrix model whose classical saddle points give rise to fuzzy spheres. Define the following matrices, for i, j, k taking values $1, 2, 3$:

$$M_k = i\epsilon_{ijk} X_i X_j - m X_k \,, \tag{5.1}$$

where the X are hermitian matrices. Now consider the action

$$S = \mathrm{tr} \vec{M}^2 \,. \tag{5.2}$$

This action can be promoted to a potential for a subsector in a supersymmetric matrix quantum mechanical system (see for example the eleven dimensional plane wave matrix

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model [39]). A different approach to obtaining an emergent fuzzy sphere from a matrix model may be found, for instance, in [40, 41].

If we set $m = 0$, classically we have an infinite family of saddles forming a continuous space of solutions. These are characterized by $[X_i, X_j] = 0$. This can be interpreted as a classical infrared divergence: the set of classical vacua is a non-compact manifold. There will be zero modes in this system and in principle the classical dynamics can run away along these directions if one adds some time dependence.

Turning on the parameter m is adding a mass term, plus some other effect linear in m. These effects lift the flat directions and one ends up with a finite set of discrete classical vacua. These vacua are characterized by matrices of order m. The general solution is described by $M_k = 0$. This can be solved by

$$X_1 = mL_1, X_2 = mL_2, X_3 = mL_3 \,, \tag{5.3}$$

for some angular momentum representation $\{L_i\}$ of the Lie algebra of SU(2). Let us choose the N dimensional irreducible representation (the spin $s = (N-1)/2$ representation).

We find in this case that the eigenvalues of X are of order N. We can evaluate our criterion for commutativity as a function of N. This is, let us compute the ratios

$$r_1 = \frac{-N\mathrm{tr}[X_1, X_2]^2}{\mathrm{tr}X_1^2\mathrm{tr}X_2^2}, \quad r_2 = \frac{\mathrm{tr}(X_1X_2X_1X_2)}{\mathrm{tr}(X_1^2X_2^2)} \,. \tag{5.4}$$

The first of these ratios is easy to compute. We get that $i[X_1, X_2] = mX_3$, and using the symmetry between X_1, X_2, X_3, we conclude that

$$r_1 = \frac{N}{\mathrm{tr}L_1^2} = \frac{3}{s(s+1)} \,. \tag{5.5}$$

When we take N to be large this expression vanishes and hence one can argue that the matrices approximately commute.

The ratio r_2 is slightly harder to evaluate. We use several identities. Firstly:

$$\mathrm{tr}[L_1, L_2]^2 = -N\frac{s(s+1)}{3} = 2\mathrm{tr}(L_1L_2L_1L_2) - 2\mathrm{tr}(L_1^2L_2^2) \,. \tag{5.6}$$

We also find by using symmetries between the X matrices that

$$N\,[s(s+1)]^2 = \mathrm{tr}L_2^2 = 3\mathrm{tr}L_1^4 + 6\mathrm{tr}(L_1^2L_2^2) \,, \tag{5.7}$$

and it is easy to show that

$$\mathrm{tr}L_1^4 = \frac{1}{15}s(1+s)(1+2s)(-1+3s+3s^2) \,. \tag{5.8}$$

With these results at hand, we compute that

$$r_2 = \frac{s^2+s-2}{s^2+s+1/2} = 1 - \frac{5}{2s(s+1)+1} \,. \tag{5.9}$$

We see that r_2 approaches one as we take N large, again suggesting this can be considered as an approximately commuting matrix model.

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In contrast to our previous examples, here the spread of the eigenvalues is of order N rather than $\sqrt{N}$. This large size can be considered as a non-perturbative effect, because there are many saddles of the matrix model. The claim is that in the large N limit these configurations of large representations go to a smooth geometry, while the vacuum where $X = 0$ is non-geometric (as we have seen before).

At this stage, one would also like to understand how to reconcile our picture of large off-diagonal masses causing commutativity with the fuzzy sphere case. Since the gauge group of the configurations is completely broken, and not $\mathrm{U}(1)^N$ as in the case of strict eigenvalue saddles, something else must replace this notion. The answer can be found in the work of Bigatti and Susskind [42]. The idea is that in the presence of some background non-commutativity, the modes that have large momentum become extended, with an extension proportional to the momentum and the noncommutativity. It is clear that here the momentum should be replaced by the angular momentum of fluctuations. We should therefore find that the fuzzy spherical harmonics correspond to stretched segments whose size grows with the angular momentum. The mass of these fluctuations also ends up being proportional to this angular momentum. This spectrum has been computed for various types of fuzzy sphere configurations (see for example [43−45]). It is clear that the modes with very high angular momentum should be very extended and massive, and for the most part the picture will not look too different from the massive off-diagonal modes that we have discussing so far in the other quantum models that we have solved. This seems to suggest that within string theory and matrix models, different notions of emergent geometry may be a lot closer than would appear at first sight.

6. Summary and discussion

The basic questions underlying this paper are as follows: suppose we are given a multi-matrix model that we cannot solve exactly. Two questions we might ask are the following. Do the degrees of freedom describing the model reduce in a strong coupling expansion from order N^2 to order N? Are these eigenvalue excitations governed by a simple (e.g. one loop) effective action?

We have found that the answer to these questions depends on the details of the model. In the cases we could solve exactly, we found that the observables

$$\frac{N\mathrm{tr}[X,Y]^2}{\mathrm{tr}X^2\mathrm{tr}Y^2} \qquad \text{and} \qquad \frac{\mathrm{tr}XYXY}{\mathrm{tr}X^2Y^2}\,, \tag{6.1}$$

captured the property of whether the model became commuting at strong coupling. The first of these observables tends to zero for all pairs of matrices in a commuting model whereas the second tends to one. We found that this occurred in a bosonic two matrix model with a commutator squared interaction. We showed that this model is described at strong coupling by an emergent two dimensional hemisphere geometry. The eigenvalue dynamics about this geometry, however, is not describable by the one loop effective action.

For the other solvable models we considered there was no emergent geometry at strong coupling. In the bosonic model of section 3.1 this occurred because integrating out the off diagonal modes induced a strong attractive force on the eigenvalues. The clumped together eigenvalues then had light off-diagonal modes connecting them. In the fermionic model of section 3.2 this attractive force was cancelled, but the lack of interactions between some of the matrices (necessary to solve the model exactly) meant that these matrices were uncorrelated and did not commute.

We then turned to fully interacting models which we cannot solve exactly. For the bosonic model we argued that there was no local emergent geometry. For models with fermions we found apparently consistent commuting saddles in a model in which the fermionic measure was constructed by hand to cancel the bosonic one loop effective action. These models appear to have an emergent spherical geometry. It would be very interesting to test our (self-consistent) results for that model with a full numerical simulation of the partition function (4.7). For the supersymmetric model with a bosonic mass term (4.14) we did not find clearcut evidence for emergent geometry. It would also be very interesting to simulate that model numerically and to compute our observables (6.1) at strong coupling.

We also considered some classical fuzzy spaces as other toy models where geometry can be argued to appear. We found that in these cases, at the classical level one also found that the matrices making up the fuzzy sphere are close to being commuting in the large N limit. The size of the matrices ends up being of order N rather than $\sqrt{N}$, and this is what makes them more classical. We also found that in these systems the notion of off-diagonal modes should be captured by the spherical harmonics of the fuzzy sphere. The angular momentum should be correlated with the size of the segment joining two putative points on the sphere.

An important feature of all the multi-matrix models (both quantum and classical) that we have considered in this paper is that they have a mass term. This is very natural in AdS/CFT, because the boundary of AdS in global coordinates contains a spatial three-sphere which gives even the zero modes a conformal mass. It is different, however, from previous considerations of matrix models as putative nonperturbative formulations of string theory, as these do not usually have mass terms. In the examples we considered, a balance between the mass term and the commutator squared interaction term was important for the emergence of local commuting geometry at strong coupling. Otherwise the coupling dependence is simply dimensional analysis. In particular, the cases with emergent geometry occur when the massless model has infrared divergences. These infrared divergences signal a potential instability of the system to grow in size. It is the regulation of this growth that seems to give us a notion of geometry. If the size is not controlled, one can imagine that the end of such a scenario is a system where all the eigenvalues have scattered at infinite distance from each other and there is nothing interesting left. We would definitely not want to call such type of configuration a geometry, but the dynamical process of reaching such an end configuration could be a cosmology of sorts. Previous cosmological applications of matrix models include [46 – 48].

We believe it is worth exploring these ideas further. There are various supersymmetric matrix quantum mechanical models where there is a mass term (for example, the plane wave matrix model [39]), and our notions of geometry might be useful to tackle the spectrum of extended objects in such systems. In a different direction, recent progress has been made connecting black hole physics with supersymmetric matrix models at finite tempera-

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ture (with infrared divergences) [49 − 51]. Infrared divergences can also be found in weakly coupled $\mathcal{N} = 4$ Super Yang-Mills theory on a three sphere, e.g. [52]. At high temperatures one expects off diagonal modes to play an important part in the dynamics. Therefore one faces an interesting challenge in combining our notion of an emergent geometry with black hole physics.

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A. Some corrections away from large $\boldsymbol{\lambda}$

A.1 Correction to the parabolic distribution

In order to compute the four point functions in section 2.2 to subleading order, we needed the leading order correction to the large λ eigenvalue distribution

$$\rho(x) = \rho_0(x) + \frac{\rho_1(x)}{\lambda^{1/3}} \,. \tag{A.1}$$

Here $\rho_0(x)$ is the parabolic distribution (2.8). It turns out that for the leading order correction we can take L to remain given by (2.9) and $\int \rho_1(x)dx = 0$, thus retaining $\int \rho(x)dx = N$.

Using (2.6) and expanding the equations of motion (2.5) to second order in strong coupling, we find the following expression for the correction to the distribution

$$\rho_1(x) = \frac{N}{L^2}\left(\frac{2}{3\pi}\right)^{1/3}\frac{3}{4\pi}\left(x\log\frac{L-x}{L+x} + L\right) . \tag{A.2}$$

Clearly this solution is somewhat formal, as the eigenvalue density diverges as $x \to \pm L$. However, inside the integral it does solve the expanded equation of motion. Furthermore, the region in which the full distribution (A.1) becomes large is exponential small ($\sim e^{-\lambda^{1/3}}$). This is related to the fact that the parabolic solution $\rho_0(x)$ was not valid very close to the endpoints, as we noted in the main text. Therefore, we should be able to use (A.2) to evaluate observables for which the integral over the eigenvalues is finite.

Using the integrals in (2.22) and (2.23) it is easy to use (A.2) to find the correction to the four point functions that is due to $\rho_1(x)$:

$$\delta(\mathrm{tr}XYXY) = \delta(\mathrm{tr}X^2Y^2) = -\frac{N^3}{40\lambda} \,. \tag{A.3}$$

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[2] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[3] G. Bhanot, U.M. Heller and H. Neuberger, *The quenched Eguchi-Kawai model*, *Phys. Lett.* **B 113** (1982) 47.

[4] D.J. Gross and Y. Kitazawa, *A quenched momentum prescription for large-N theories*, *Nucl. Phys.* **B 206** (1982) 440.

[5] T. Eguchi and H. Kawai, *Reduction of dynamical degrees of freedom in the large-N gauge theory*, *Phys. Rev. Lett.* **48** (1982) 1063.

[6] A. Gonzalez-Arroyo and M. Okawa, *A twisted model for large-N lattice gauge theory*, *Phys. Lett.* **B 120** (1983) 174.

[7] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large-N limit of theories with sixteen supercharges*, *Phys. Rev.* **D 58** (1998) 046004 [hep-th/9802042].

[8] E.P. Wigner, *Characteristics vectors of bordered matrices with infinite dimensions II*, *Ann. Math.* **65** (1957) 203.

[9] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar diagrams*, *Commun. Math. Phys.* **59** (1978) 35.

[10] L.L. Chau and Y. Yu, *Unitary polynomials in normal matrix models and Laughlin's wave functions for the fractional quantum Hall effect*, *Phys. Lett.* **A 167** (1992) 452 [UCDPHYS-PUB-91-13].

[11] R. Teodorescu, E. Bettelheim, O. Agam, A. Zabrodin and P. Wiegmann, *Normal random matrix ensemble as a growth problem: evolution of the spectral curve*, *Nucl. Phys.* **B 704** (2005) 407 [hep-th/0401165].

[12] L.-L. Chau and O. Zaboronsky, *On the structure of correlation functions in the normal matrix model*, *Commun. Math. Phys.* **196** (1998) 203 [hep-th/9711091].

[13] C. Itzykson and J.B. Zuber, *The planar approximation. 2*, *J. Math. Phys.* **21** (1980) 411.

[14] W. Krauth, H. Nicolai and M. Staudacher, *Monte Carlo approach to M-theory*, *Phys. Lett.* **B 431** (1998) 31 [hep-th/9803117].

[15] W. Krauth and M. Staudacher, *Finite Yang-Mills integrals*, *Phys. Lett.* **B 435** (1998) 350 [hep-th/9804199].

[16] W. Krauth and M. Staudacher, *Eigenvalue distributions in Yang-Mills integrals*, *Phys. Lett.* **B 453** (1999) 253 [hep-th/9902113].

[17] W. Krauth, J. Plefka and M. Staudacher, *Yang-Mills integrals*, *Class. and Quant. Grav.* **17** (2000) 1171 [hep-th/9911170].

[18] T. Hotta, J. Nishimura and A. Tsuchiya, *Dynamical aspects of large-N reduced models*, *Nucl. Phys.* **B 545** (1999) 543 [hep-th/9811220].

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[19] J. Ambjørn, K.N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Large-N dynamics of dimensionally reduced 4D SU(N) super Yang-Mills theory*, *JHEP* **07** (2000) 013 [hep-th/0003208].

[20] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instanton calculus and the AdS/CFT correspondence in $N = 4$ superconformal field theory*, *Nucl. Phys.* **B 552** (1999) 88 [hep-th/9901128].

[21] S.-J. Rey and Y. Hikida, *Black hole as emergent holographic geometry of weakly interacting hot Yang-Mills gas*, *JHEP* **08** (2006) 051 [hep-th/0507082].

[22] H.-Y. Chen and D. Tong, *Instantons and emergent $AdS_3 \times S^3$ geometry*, *JHEP* **06** (2006) 017 [hep-th/0604090].

[23] D. Berenstein, *Large-N BPS states and emergent quantum gravity*, *JHEP* **01** (2006) 125 [hep-th/0507203].

[24] D. Berenstein and D.H. Correa, *Emergent geometry from q-deformations of $N = 4$ super Yang-Mills*, *JHEP* **08** (2006) 006 [hep-th/0511104].

[25] D. Berenstein, *Strings on conifolds from strong coupling dynamics, part I*, *JHEP* **04** (2008) 002 [arXiv:0710.2086].

[26] D.E. Berenstein and S.A. Hartnoll, *Strings on conifolds from strong coupling dynamics: quantitative results*, *JHEP* **03** (2008) 072 [arXiv:0711.3026].

[27] D. Berenstein, D.H. Correa and S.E. Vazquez, *All loop BMN state energies from matrices*, *JHEP* **02** (2006) 048 [hep-th/0509015].

[28] D. Berenstein, R. Cotta and R. Leonardi, *Numerical tests of AdS/CFT at strong coupling*, *Phys. Rev.* **D 78** (2008) 025008 [arXiv:0801.2739].

[29] V.A. Kazakov, I.K. Kostov and N.A. Nekrasov, *D-particles, matrix integrals and KP hierarchy*, *Nucl. Phys.* **B 557** (1999) 413 [hep-th/9810035].

[30] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[31] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring*, *Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115].

[32] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Matrix string theory*, *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030].

[33] U. Gursoy, S.A. Hartnoll, T.J. Hollowood and S.P. Kumar, *Topology change and new phases in thermal $N = 4$ SYM theory*, *JHEP* **11** (2007) 020 [hep-th/0703100].

[34] O. Aharony and S.A. Hartnoll, *A phase transition in commuting Gaussian multi-matrix models*, arXiv:0706.2861.

[35] P. Austing and J.F. Wheater, *The convergence of Yang-Mills integrals*, *JHEP* **02** (2001) 028 [hep-th/0101071].

[36] P. Austing and J.F. Wheater, *Convergent Yang-Mills matrix theories*, *JHEP* **04** (2001) 019 [hep-th/0103159].

[37] R.C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022 [hep-th/9910053].

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[38] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, hep-th/0003136.

[39] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [hep-th/0202021].

[40] H. Steinacker, *Quantized gauge theory on the fuzzy sphere as random matrix model*, *Nucl. Phys.* **B 679** (2004) 66 [hep-th/0307075].

[41] R. Delgadillo-Blando, D. O'Connor and B. Ydri, *Geometry in transition: a model of emergent geometry*, *Phys. Rev. Lett.* **100** (2008) 201601 [arXiv:0712.3011].

[42] D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, *Phys. Rev.* **D 62** (2000) 066004 [hep-th/9908056].

[43] D. Berenstein and R. Corrado, *Matrix theory on ALE spaces and wrapped membranes*, *Nucl. Phys.* **B 529** (1998) 225 [hep-th/9803048].

[44] G. Ishiki, S. Shimasaki, Y. Takayama and A. Tsuchiya, *Embedding of theories with* SU(2|4) *symmetry into the plane wave matrix model*, *JHEP* **11** (2006) 089 [hep-th/0610038].

[45] H. Kaneko, Y. Kitazawa and K. Matsumoto, *Effective actions of IIB matrix model on S^3*, *Phys. Rev.* **D 76** (2007) 084024 [arXiv:0706.1708].

[46] D.Z. Freedman, G.W. Gibbons and M. Schnabl, *Matrix cosmology*, *AIP Conf. Proc.* **743** (2005) 286 [hep-th/0411119].

[47] B. Craps, S. Sethi and E.P. Verlinde, *A matrix big bang*, *JHEP* **10** (2005) 005 [hep-th/0506180].

[48] J. Erdmenger, R. Meyer and J.-H. Park, *Spacetime emergence in the Robertson-Walker universe from a matrix model*, *Phys. Rev. Lett.* **98** (2007) 261301 [arXiv:0705.1586].

[49] S. Catterall and T. Wiseman, *Towards lattice simulation of the gauge theory duals to black holes and hot strings*, *JHEP* **12** (2007) 104 [arXiv:0706.3518].

[50] K.N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, *Monte Carlo studies of supersymmetric matrix quantum mechanics with sixteen supercharges at finite temperature*, *Phys. Rev. Lett.* **100** (2008) 021601 [arXiv:0707.4454].

[51] S. Catterall and T. Wiseman, *Black hole thermodynamics from simulations of lattice Yang-Mills theory*, *Phys. Rev.* **D 78** (2008) 041502 [arXiv:0803.4273].

[52] T.J. Hollowood, S.P. Kumar, A. Naqvi and P. Wild, *$N = 4$ SYM on S^3 with near critical chemical potentials*, *JHEP* **08** (2008) 046 [arXiv:0803.2822].